

UNIVERSITY OF ILLINOIS  
AT URBANA-CHAMPAIGN

# *Pulses in transmission lines*

Physics 401, Spring 2013

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[illinois.edu](http://illinois.edu)

# Transmission lines.

## Agenda.

- Definition
- Distributed parameters network
- Pulses in transmission line
- Wave equation and wave propagation
- Reflections. Resistive load
- Thévenin's theorem
- Reflection. Non resistive load
- Appendix. Error propagation



# Transmission lines. Distributed parameters network.

- Transmission line is a specialized cable designed to carry alternating current of radio frequency, that is, currents with a frequency high enough that its wave nature must be taken into account.



Courtesy Wikipedia

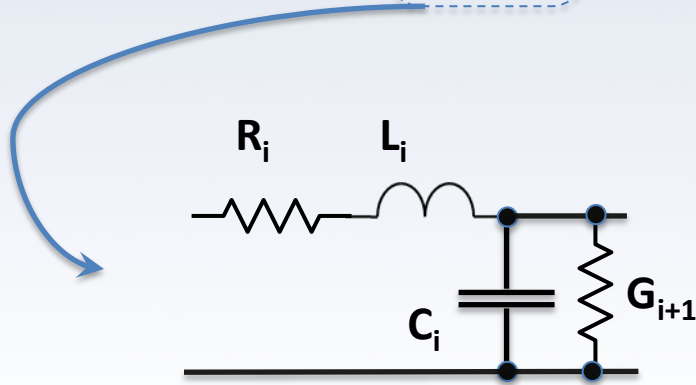
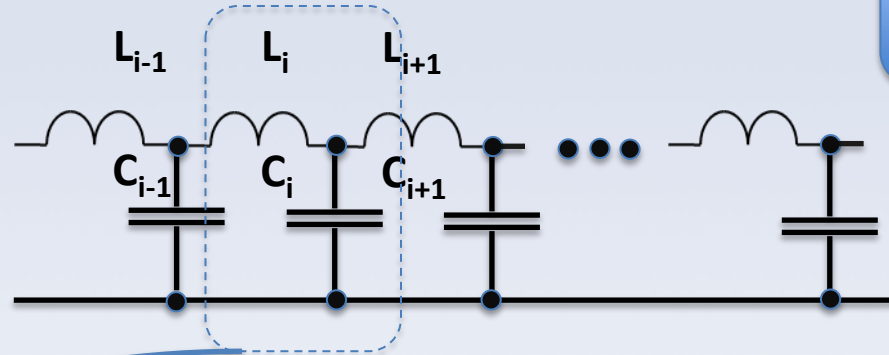


# Transmission lines. Distributed parameters network

Simplified equivalent circuit



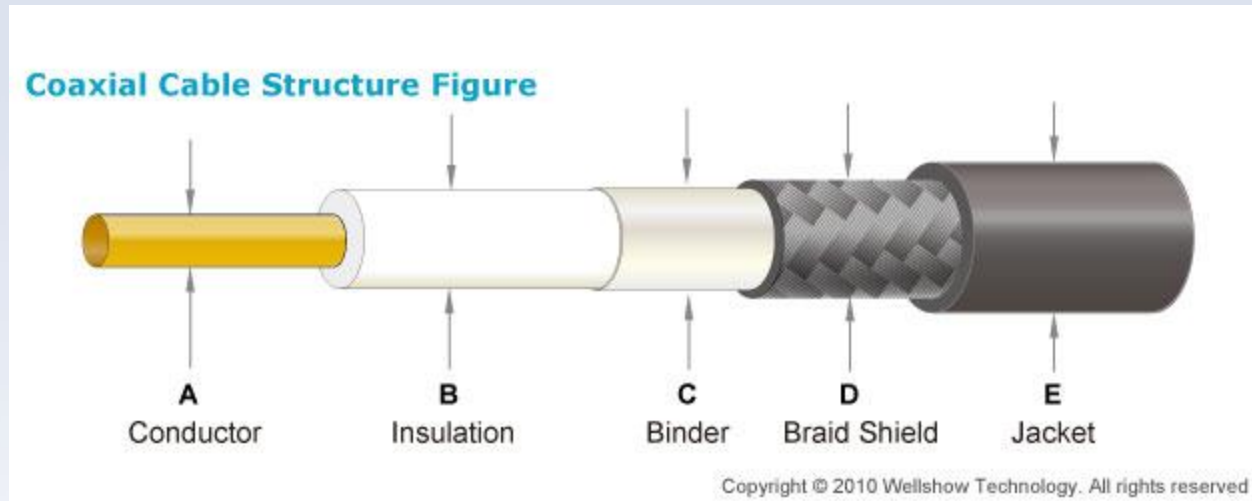
Ideal case



Real situation



# Coaxial cable



## Specification:

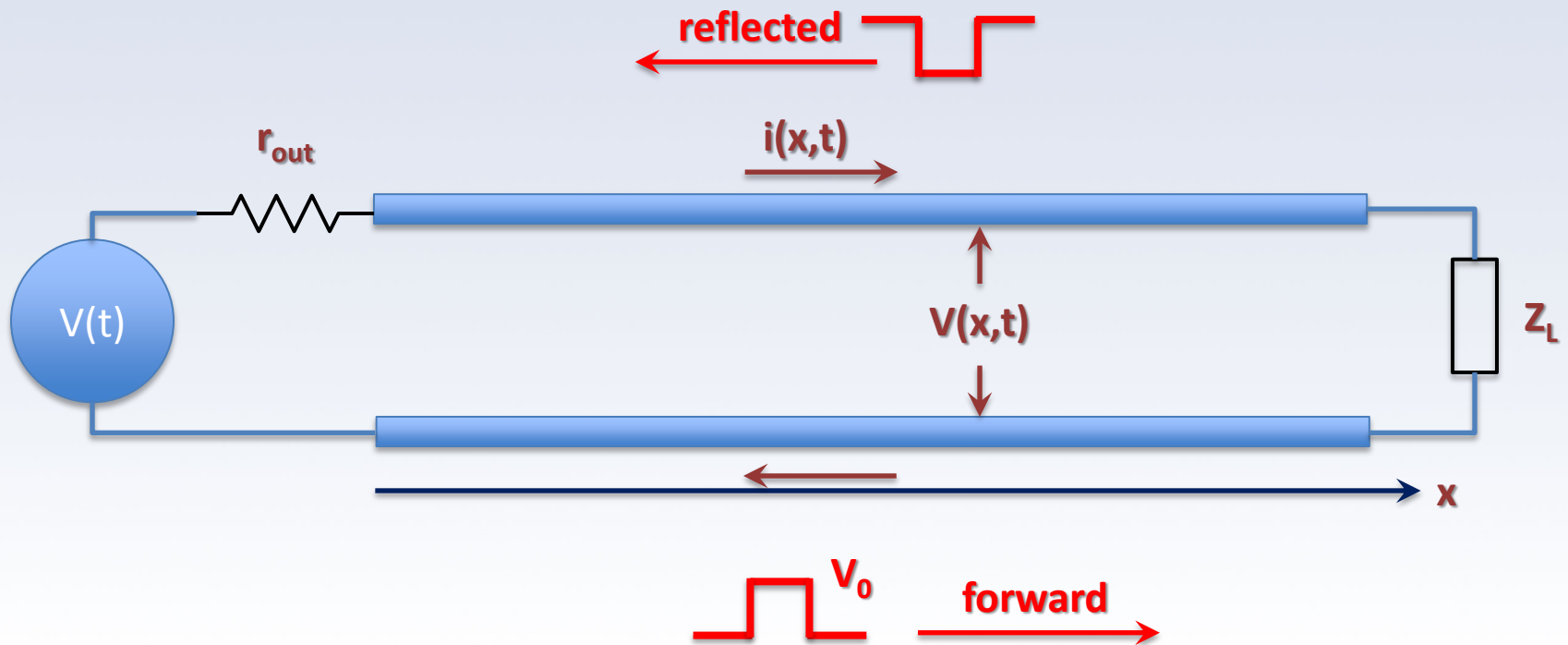
**Impedance: 53  $\Omega$**

**Capacitance: 83 pF/m**

**Conductor: Bare Copper Wire (1/1.02mm)**



# Pulses in transmission line



# Setup

Wavetek 81

Tektronix 3012B



Sync output

Triggering input

Signal output

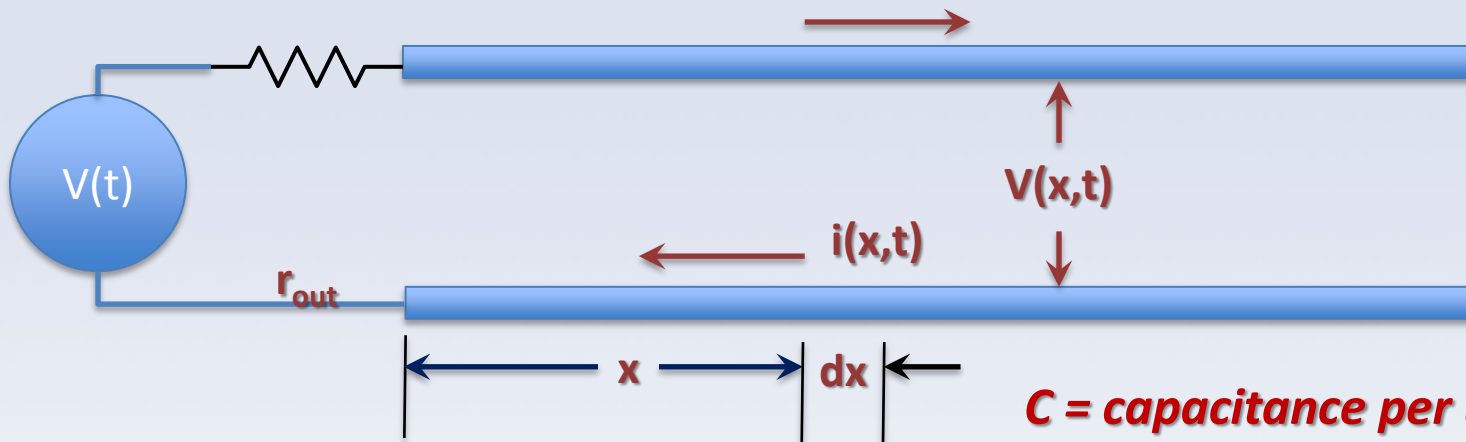
RG8U



Load



# The Wave Equation



*C = capacitance per unit length*  
*L = inductance per unit length*

$$CdxV = -dq;$$

$$C \frac{\partial V}{\partial t} = -\frac{\partial q}{\partial t} = i;$$

$$\frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t}$$

$$dV = -(Ldx) \frac{di}{dt};$$

$$\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t}$$



# The Wave Equation

$$\frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t}$$

$$\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t}$$

$$\frac{\partial}{\partial t}$$

$$\frac{\partial^2 i}{\partial t \partial x} = -C \frac{\partial^2 V}{\partial t^2} \quad (1)$$

$$\frac{\partial}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2} = -L \frac{\partial^2 i}{\partial x \partial t} \quad (2)$$

Combining (1) and (2)

$$\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$



# The Wave Equation.

## Voltage and current waves.

$$\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} \quad \frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$

Looking for solution

$$V(x,t) = V_0 \sin \omega \left( t - \frac{x}{v} \right)$$

$$i(x,t) = i_0 \sin \omega \left( t - \frac{x}{v} \right)$$

Now substituting  $V(x,t)$  and  $i(x,t)$  in

$$\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t} \quad \frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t}$$

We can find  $V_0 = i_0 \sqrt{\frac{L}{C}}$  or

$$v = \frac{1}{\sqrt{LC}}$$

Speed of wave propagation

$$V(x,t) = \sqrt{\frac{L}{C}} i(x,t) = Z_k i(x,t)$$

$Z_k$  - characteristic Impedance

Equivalent to Ohm's law equation



# Characteristic impedance

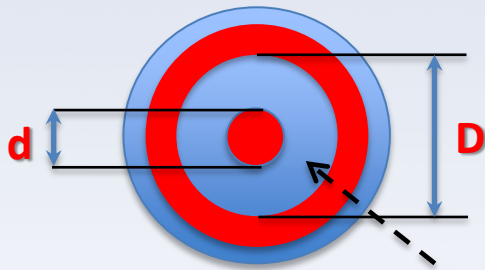
$$Z_k = \sqrt{\frac{L}{C}}$$

$C$  = capacitance per unit length  
 $L$  = inductance per unit length

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ (F/m)}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}$$

Cross-section of the coaxial cable



$$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln\left(\frac{D}{d}\right)} \text{ (F/m)} \quad L = \frac{\mu_0\mu_r}{2\pi} \ln\left(\frac{D}{d}\right) \text{ (H/m)}$$

$\epsilon_r$  – dielectric permittivity  
 $\mu_r$  – magnetic permeability  $\approx 1$

Finally for coaxial cable: 
$$Z_k = \frac{138}{\sqrt{\epsilon_r}} \log\left(\frac{D}{d}\right) \text{ (Ohms)}$$



# Speed of wave propagation, delay.

$$v = \frac{1}{\sqrt{LC}}$$

Speed of wave propagation



$$v = \frac{1}{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}} = \frac{c}{\sqrt{\mu_r \epsilon_r}} \approx \frac{c}{\sqrt{\epsilon_r}}$$

$\approx 1$

For polyethylene  $\epsilon_r \sim 2.25$  (up to 1GHz)

$$\text{Delay time } \tau = \frac{1}{v} (\text{s/m}) \approx 3.336 \cdot 10^{-9} \sqrt{\epsilon_r} (\text{s/m}) = 3.336 \sqrt{\epsilon_r} (\text{ns/m})$$

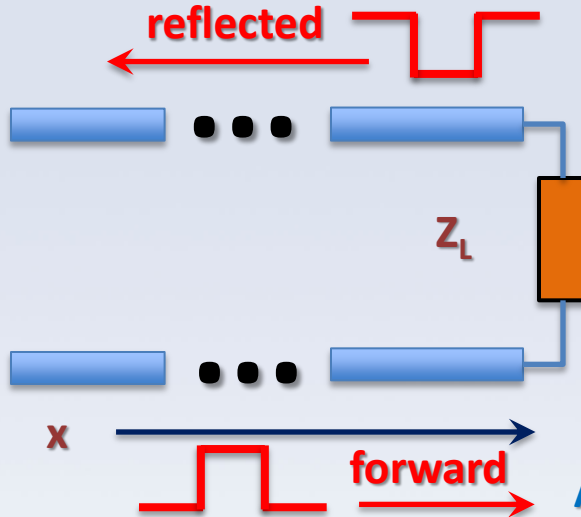
**RG-8/U,**  
**RG58U:**

Inner Insulation Materials: Polyethylene  
Nominal Impedance: 52 ohm  
Delay time  $\sim 5\text{ns/m}$



# Reflection in transmission line

solution for the traveling in opposite direction



$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}$$

$$V(x,t) = V_0 \sin \omega \left( t - \frac{x}{v} \right)$$

$$i(x,t) = i_0 \sin \omega \left( t - \frac{x}{v} \right)$$

For reflected wave  $V_r = -Z_k i_r$

At any point of the transmission line:

1. Resistive load  $Z_L = R_L$

$$\frac{V}{i} = R_L$$

$$V = V_r + V_i$$

$$i = i_r + i_i = \frac{V_i}{Z_k} - \frac{V_r}{Z_k}$$

$$\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$$



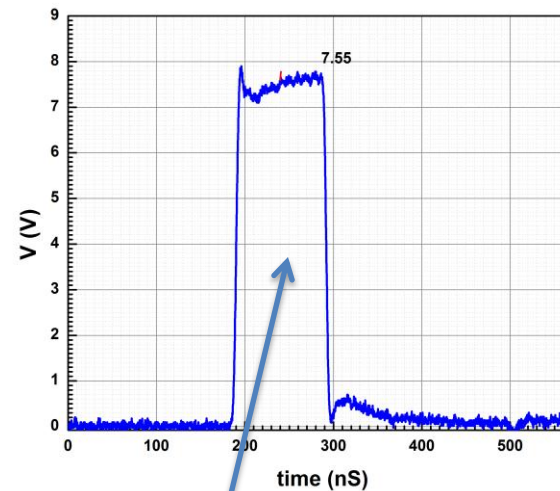
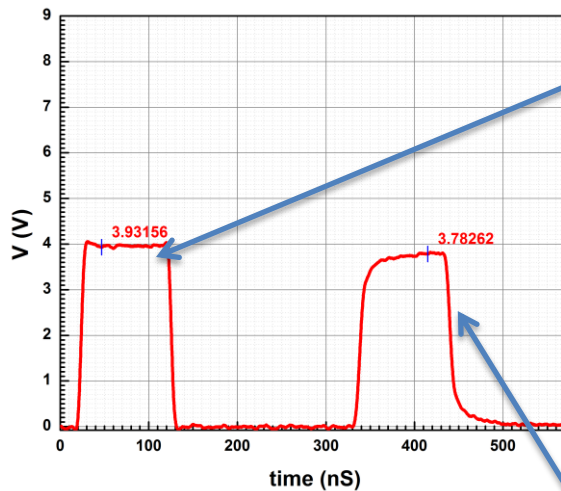
# Reflection in transmission line

Resistive load  $Z_L = R_L$

$$\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$$

Open line  $R_L = \infty \rightarrow V_r = V_i$

Incident pulse



Reflected pulse

End of the line



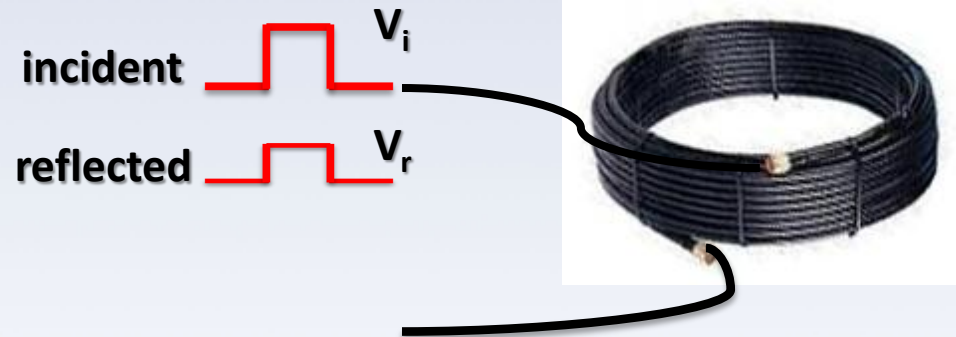
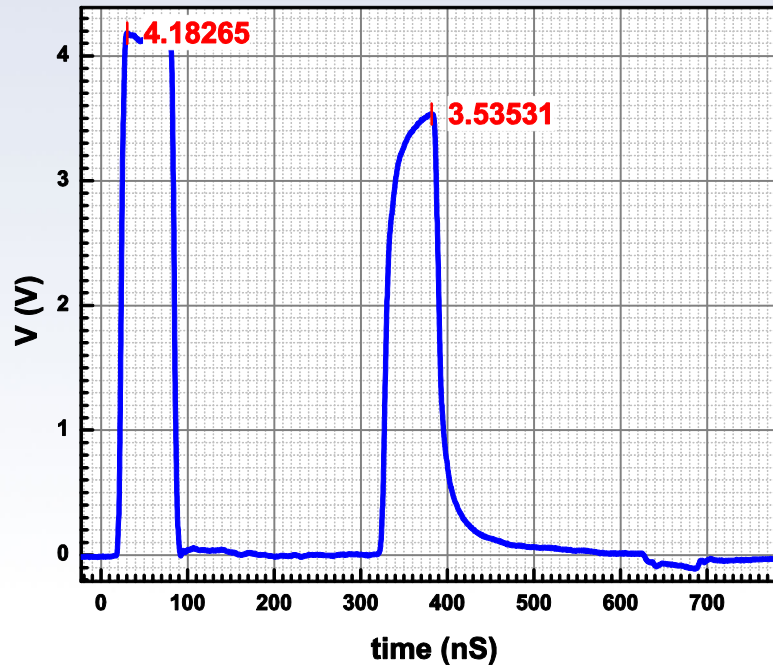
# Reflection in transmission line. Loses.

Theory:  $R_L = \infty \rightarrow V_r = V_i$

Attenuation (dB per 100 feet)

| MHz    | 30  | 50  | 100 | 146 | 150 |
|--------|-----|-----|-----|-----|-----|
| RG-58U | 2.5 | 4.1 | 5.3 | 6.1 | 6.1 |

Experiment RG 58U

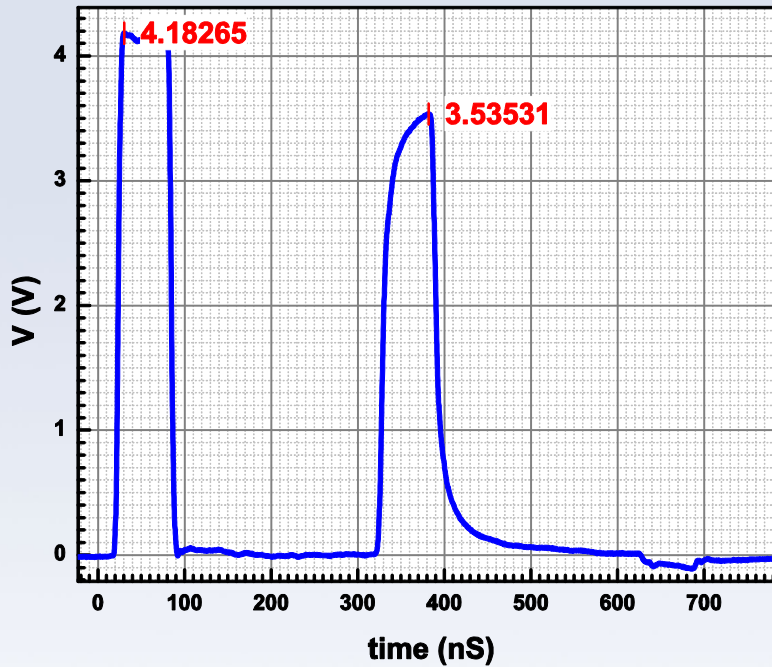


$$ATTN(db) = 20 \log \left( \frac{V_i}{V_r} \right)$$

Important parameter for cable is attenuation per length

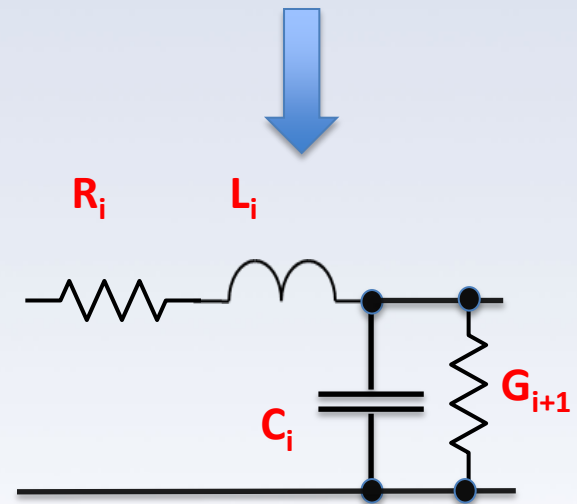


# Reflection in transmission line. Loses.

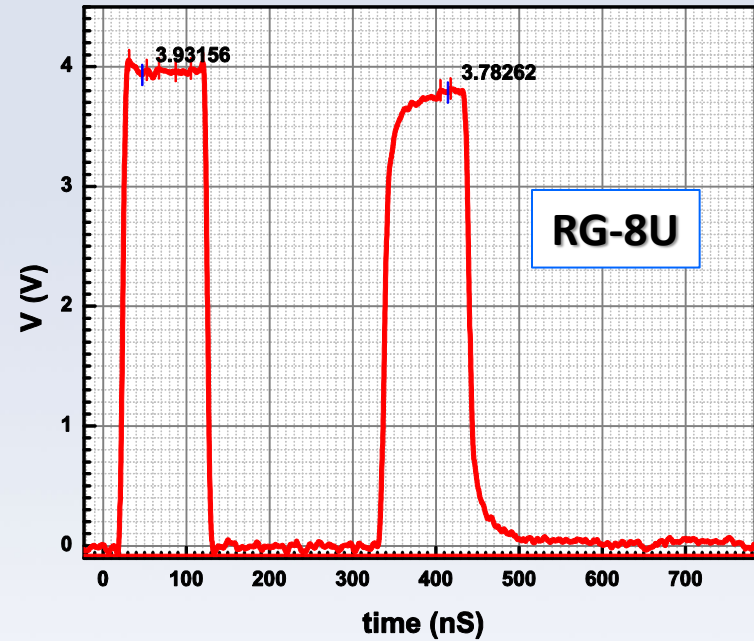
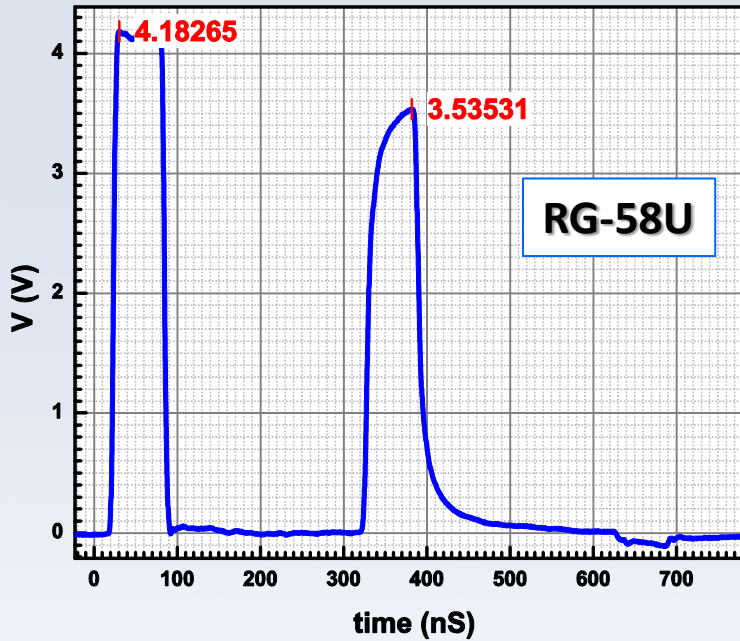


In our case:  $Attn(200\text{ ft}) = 20\log\left(\frac{4.18}{3.54}\right) \approx 1.46\text{ dB}$

Where it is coming from?



# Different cables loses.

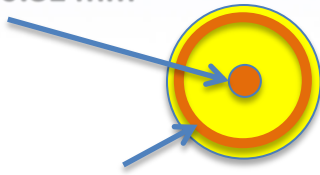


$$Attn(200\ ft) = 20 \log \left( \frac{4.18}{3.54} \right) \approx 1.46\ dB$$

>

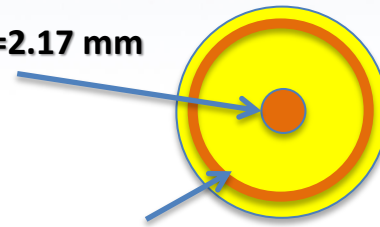
$$Attn(200\ ft) = 20 \log \left( \frac{3.932}{3.78} \right) \approx 0.335\ dB$$

Core  $\phi=0.81\ mm$



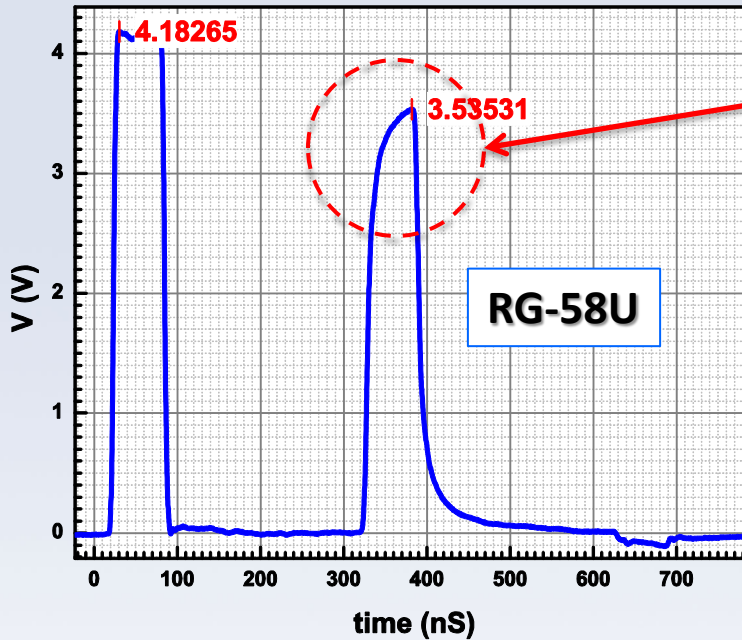
Dielectric  $\phi=2.9\ mm$

Core  $\phi=2.17\ mm$



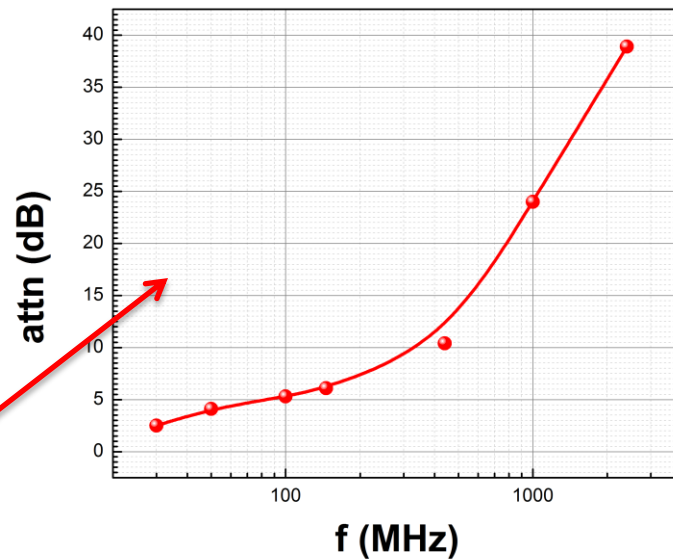
Dielectric  $\phi=7.2\ mm$

# Loses. Frequency dispersion.

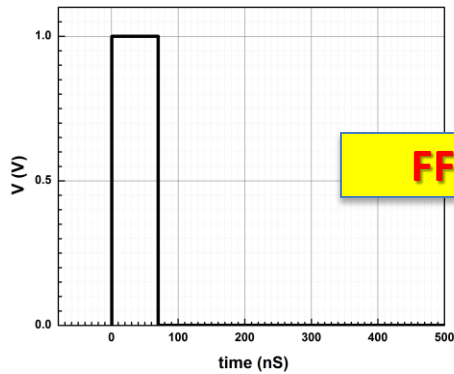


Reflected pulse does not follow the shape of the incident pulse

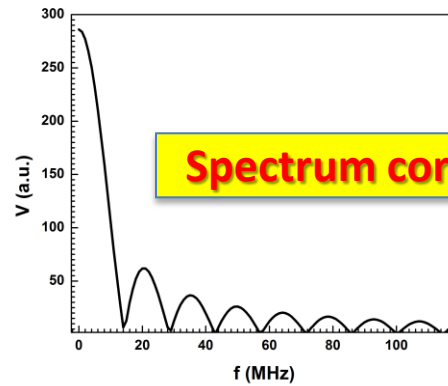
Frequency dependence of the attenuation RG-58U cable



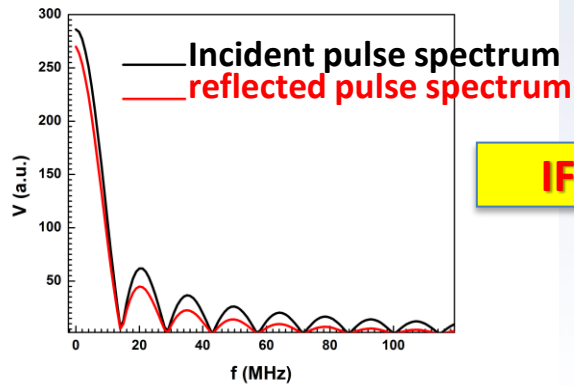
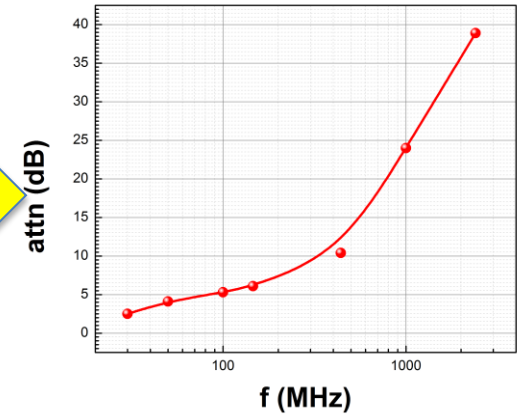
# Loses. Frequency dispersion.



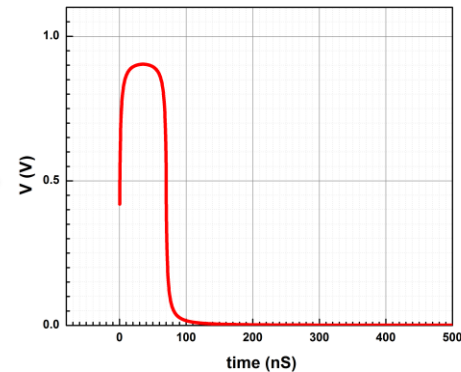
FFT



Spectrum correction



IFFT

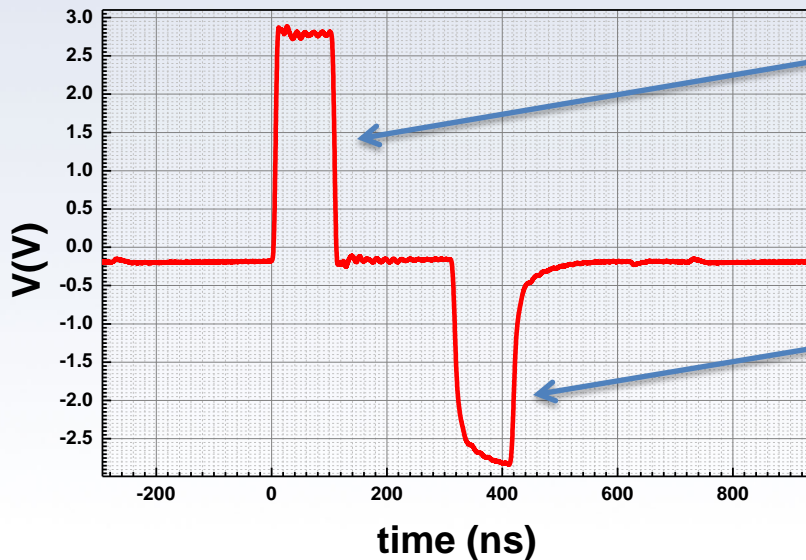


# Reflection in transmission line

Resistive load  $Z_L = R_L$

$$\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$$

Shorted line  $R_L = 0 \rightarrow V_r = -V_i$



Incident pulse

Reflected pulse



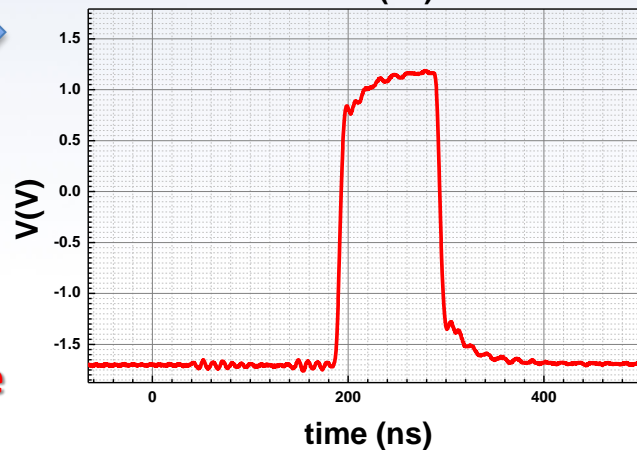
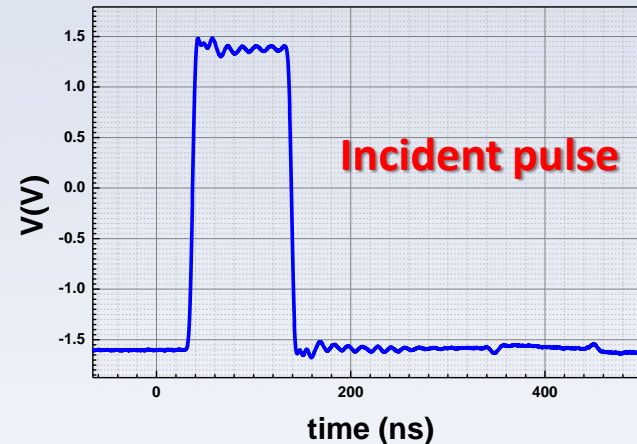
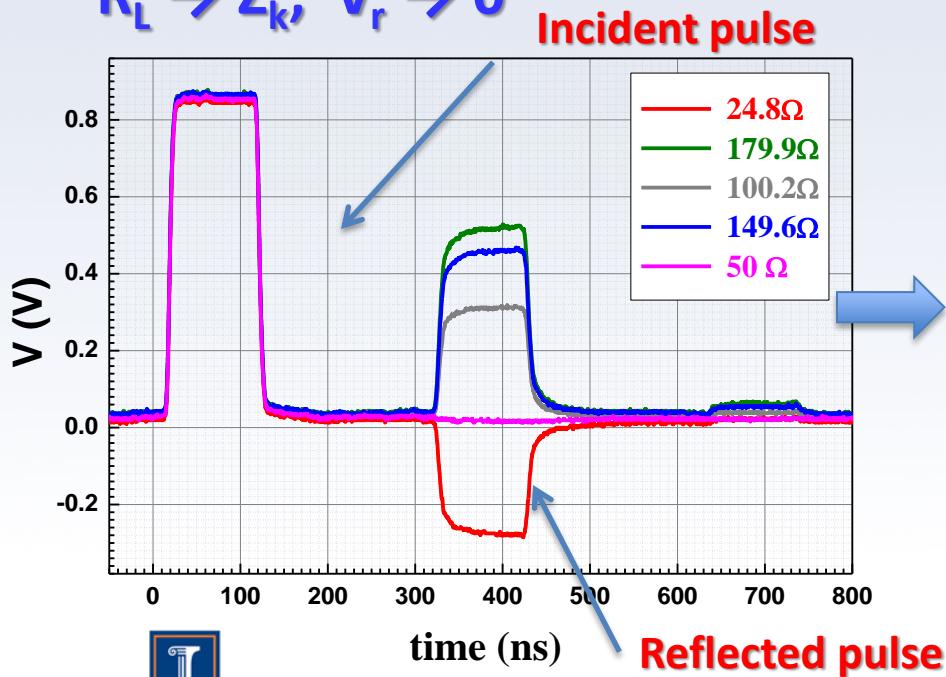
# Reflection in transmission line.

Resistive load  $Z_L = R_L$

$$\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$$

Matching the load impedance

$R_L \Rightarrow Z_k; V_r \rightarrow 0$



# Thévenin's theorem

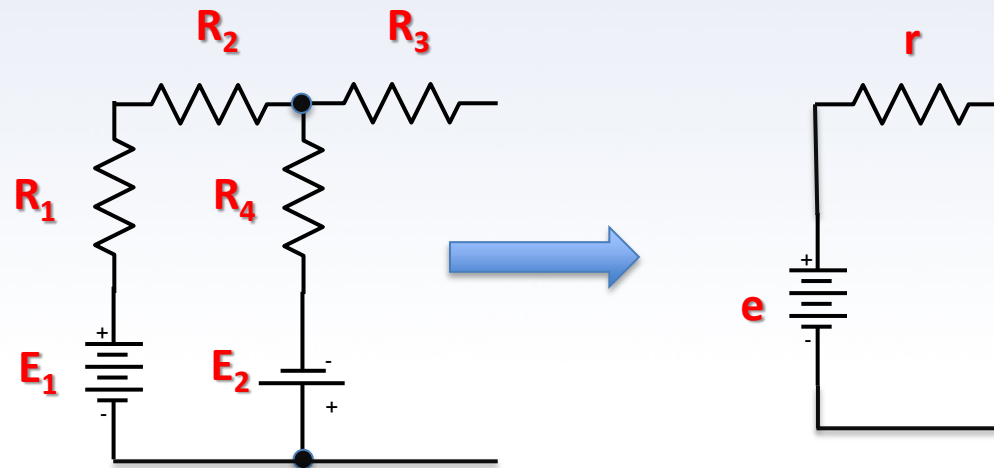


Hermann Ludwig  
Ferdinand von Helmholtz  
(1821-1894)



Léon Charles Thévenin  
(1857-1926)

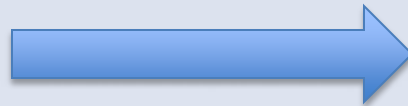
Any combination of batteries and resistances with two terminals can be replaced by a single voltage source  $e$  and a single series resistor  $r$



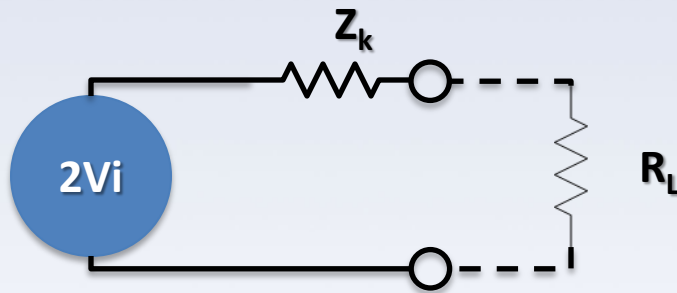
# Thévenin's theorem. Transmission line.

$$V = V_r + V_i = i \bullet R_L$$

$$i = i_r + i_i = \frac{V_i}{Z_k} - \frac{V_r}{Z_k}$$



$$i = \frac{2V_i}{R_L + Z_k}$$



From this equivalent equation we can find the maximum possible power delivered to  $R_L$

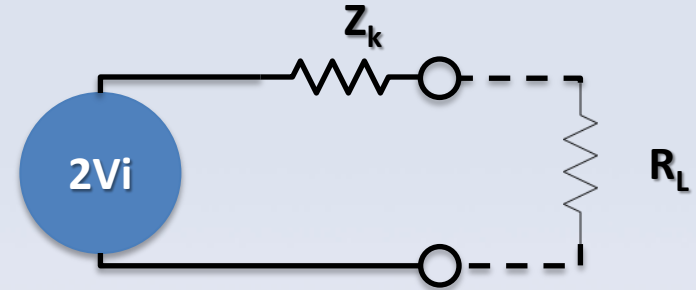
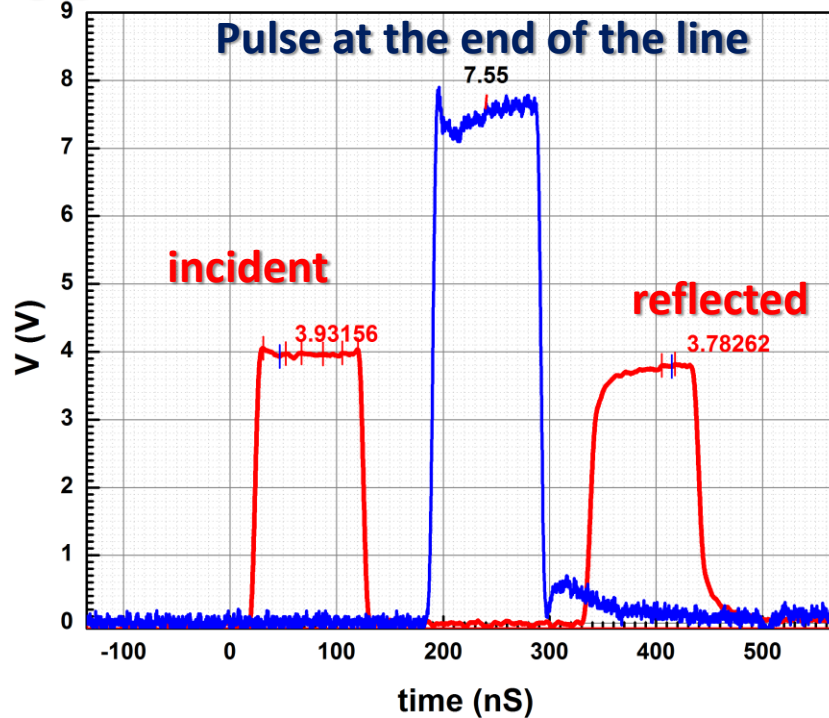
$$P = i^2 R_L = \frac{(2V_i)^2}{(R_L + Z)^2} R_L$$

**$P = P_{\max}$  if  $R_L = Z_k$  (no reflection)**



# Thévenin's theorem. Experiment.

RG 8U



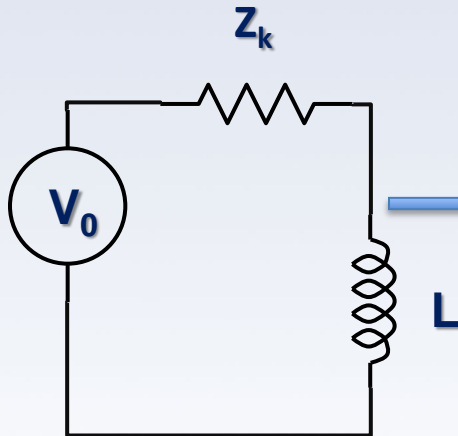
This experiment better to perform on RG 8U cable because of lower attenuation

$R_L = \infty$ , amplitude of the pulse at the end of line is expected to be  $2V_i$ , where  $V_i$  is the amplitude of the incident pulse



# Reflection. Inductive load.

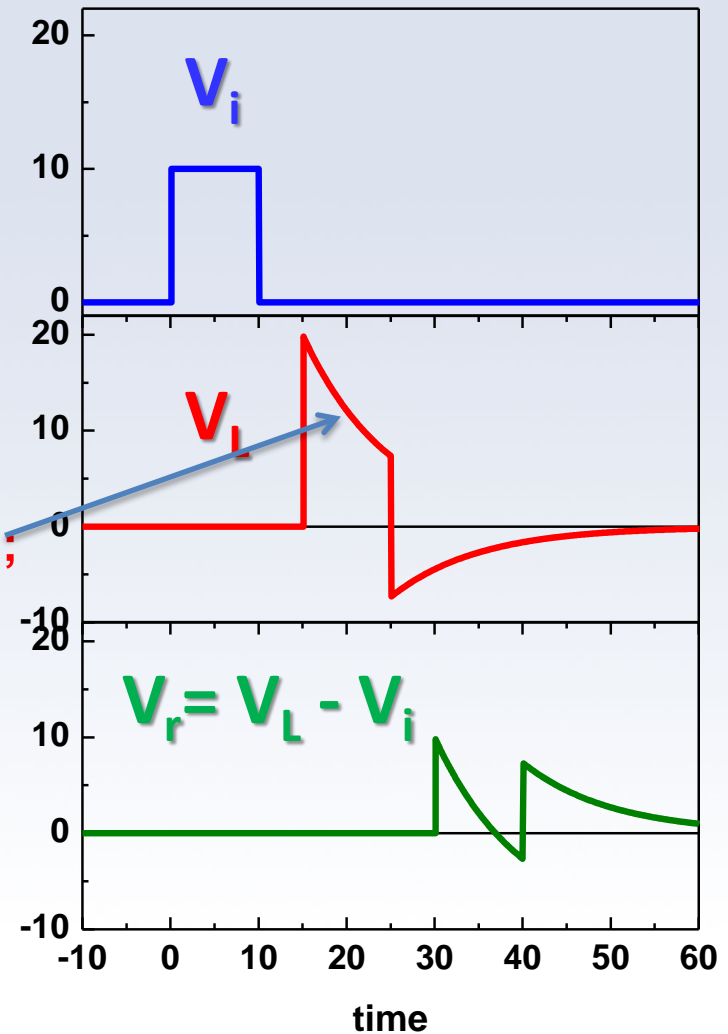
$$i = \frac{2V_i}{Z_L + Z_k}$$



$$2V_i = iZ_k - L \frac{di}{dt};$$

$$i = i_0 \left( 1 - \exp\left(\frac{-t}{\tau}\right) \right);$$

$$\tau = \frac{L}{Z_k}$$

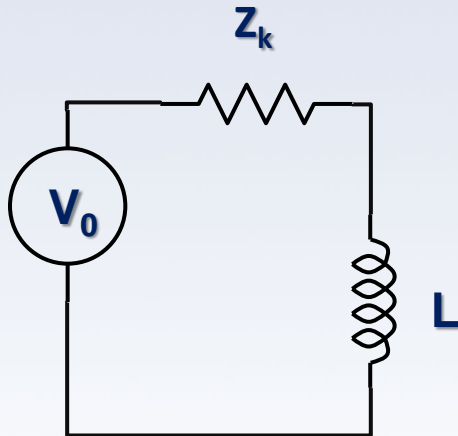


# Reflection. Inductive load. Experiment

$$i = \frac{2V_i}{Z_L + Z_k}$$

$$2V_i = iZ_k - L \frac{di}{dt};$$

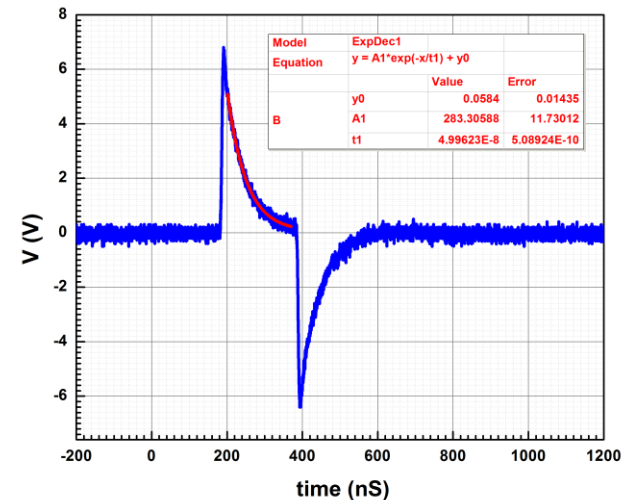
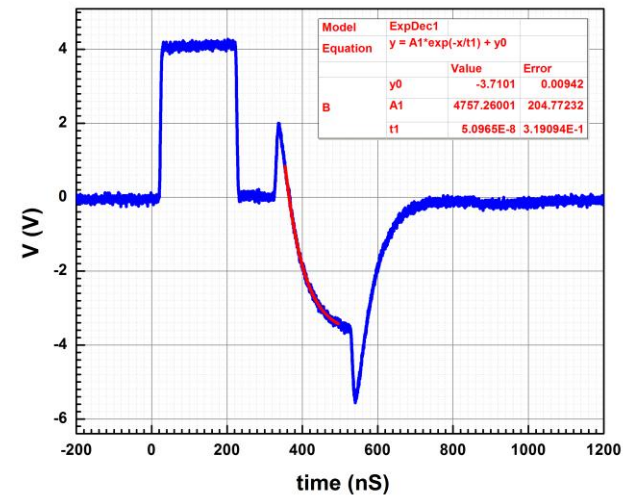
$$i = i_0 \left( 1 - \exp\left(\frac{-t}{\tau}\right) \right);$$



$$\tau = \frac{L}{Z_k}$$

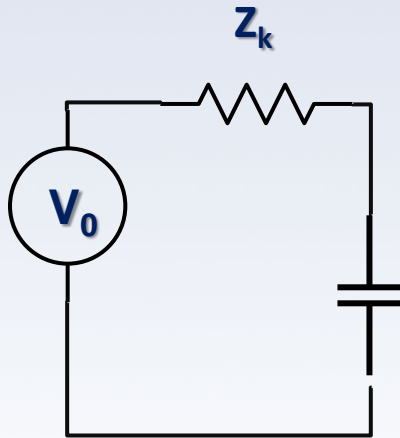
$$\tau \cong 50\text{ns},$$

$$L = \tau Z_k \sim 2.5\mu\text{H}$$



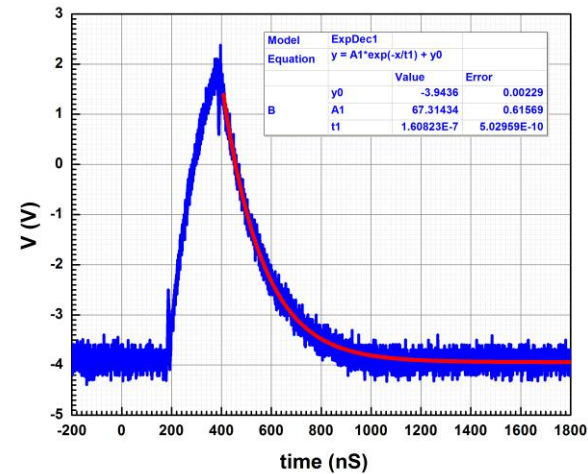
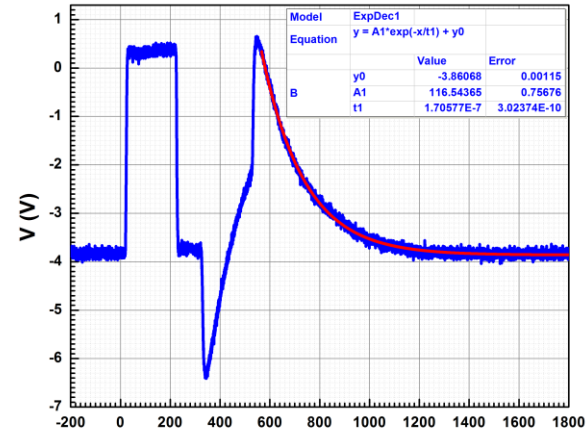
# Reflection. Capacitive load. Experiment

$$i = \frac{2V_i}{Z_L + Z_k}$$



$$\tau = Z_k C$$

$$C = \frac{\tau}{Z_k} \approx 3.2 \text{ nF}$$



# Appendix #1.

# Export graphs from Origin

OriginPro 8.6 (Academic) 64-bit - E:\Teaching\P401\Eugene lectures Sp

File Edit View Graph Data Analysis Gadgets Tools Fc

New  
Open... Ctrl+O  
Open Excel... Ctrl+E  
Open Sample OPJ  
Append...  
Close  
Save Project Ctrl+S  
Save Project As...  
Save Window As...  
Save Template As...  
Save Project As Analysis Template...  
Print... Ctrl+P  
Print Preview  
Page Setup...  
Import  
Export Graphs  
Batch Processing...  
Recent Imports  
Recent Exports  
Recent Books  
Recent Graphs  
Recent Projects  
Exit

1 <Last used>  
Open Dialog...

Import and Export: expGraph

Dialog Theme

Description Export graph(s) to graphics file(s)

Image Type Portable Network Graphics (\*.png)

Export Active Page

File Name(s) <long name>

Path s:\kolla\Documents\OriginLab\86\User Files\

Overwrite Existing Ask

Graph Theme <Original>

Export Settings  
Use Current Speed Mode Display for Export Apply Page Setting  
Margin Control Page  
Clip Border Width 5

Image Size

Preview Preview Apply OK Cancel

No Preview

Check the "Auto Preview" checkbox to display updated preview, and click "Preview" button when needed.

Encapsulated Postscript (\*.eps)  
Adobe Illustrator (\*.ai)  
Bitmap (\*.bmp)  
Enhanced MetaFile (\*.emf)  
Encapsulated Postscript (\*.eps)  
Graphics Interchange Format (\*.gif)  
Joint Photographic Experts Group (\*.jpg)  
Zsoft PC Paintbrush Bitmap (\*.pcx)  
Portable Document Format (\*.pdf)  
Portable Network Graphics (\*.png)  
Adobe Photoshop (\*.psd)  
Truevision Targa (\*.tga)  
Tag Image File (\*.tif)  
Windows MetaFile (\*.wmf)



# Appendix #2.

# Reminders

1. The reports should be uploaded to the *proper* folder and *only* to the proper folder

For example folder *RLC Lab Report\_L1* should be used by students from L1 section only

I would recommend the file name style as:

**L1\_lab2\_student1**

Lab section

Lab number

Your name

2. Origin template for this week Lab:

[\\Phyapportal\PHYCS401\Common\Origin templates\Transmission line\Time trace.otp](#)



# Appendix #3. RLC resonance circuit. Fitting.

$R_{\text{limit}}$  — a series to LCR circuit resistor

Fitting parameters:

(1)  $\frac{R}{R_{\text{limit}}}$ , (2)  $\omega_0 = \frac{1}{\sqrt{LC}}$ ,

(3)  $Q = \frac{1}{R} \sqrt{\frac{L}{C}}$

Initial values can be estimated from circuit parameters.

$$\tilde{H} = \left( \frac{1}{R_{\text{limit}}} \right) \frac{R + j\omega L}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j\omega RC} =$$

$$\left( \frac{R}{R_{\text{limit}}} \right) \frac{1 + j \frac{\omega}{\omega_0} Q}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j \frac{\omega}{\omega_0} \frac{1}{Q}}$$

$$= \left( \frac{R}{R_{\text{limit}}} \right) \frac{1 - j \frac{\omega}{\omega_0} \left( \frac{1}{Q} - Q \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \right)}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 \frac{1}{Q^2}}$$

For fitting results and actual fitting function equation go to:

[Phyapportal\PHYCS401\Common\SimpleExamples\Lab 3 Frequency Domain Analysis\\_example.opj](http://Phyapportal\PHYCS401\Common\SimpleExamples\Lab 3 Frequency Domain Analysis_example.opj)



Now to simplify the equation we can introduce the reduced frequency  $\gamma = \frac{\omega}{\omega_0}$  and transfer function components can be presented as:

$$\tilde{H}(\omega) = \left( \frac{R}{R_0} \bullet Q \right) \frac{Q - j\gamma \left( 1 - Q^2 (1 - \gamma^2) \right)}{Q^2 (1 - \gamma^2)^2 + (\gamma)^2};$$

$$H_{RE} = \left( \frac{R}{R_0} \bullet Q \right) \frac{Q}{Q^2 (1 - \gamma^2)^2 + (\gamma)^2};$$

$$H_{IM} = \left( \frac{R}{R_0} \bullet Q \right) \frac{-j\gamma \left( 1 - Q^2 (1 - \gamma^2) \right)}{Q^2 (1 - \gamma^2)^2 + (\gamma)^2}$$



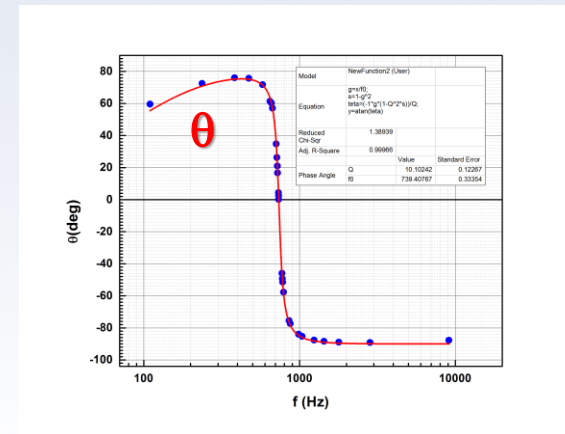
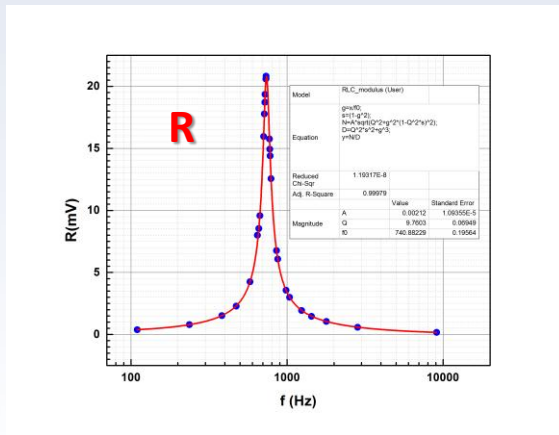
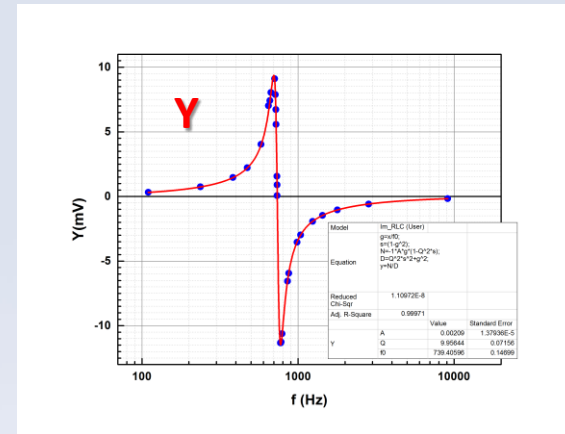
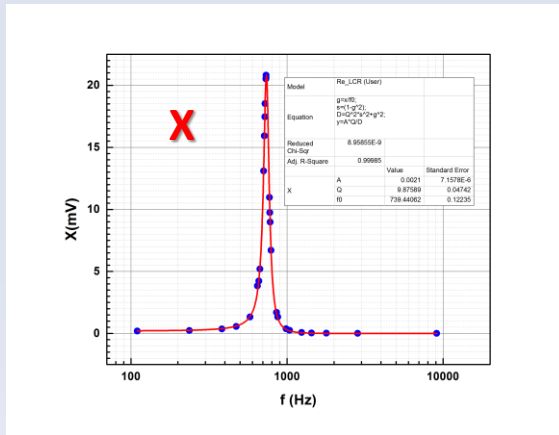
Two other components the modulus and the phase shift:

$$|\tilde{H}| = \left( \frac{R}{R_0} \cdot Q \right) \frac{\sqrt{Q^2 + 2(1 - Q^2(1 - \gamma^2))}}{Q^2(1 - \gamma^2) + (\gamma)^2};$$

$$\theta = \arctan\left(\frac{H_{IM}}{H_{RE}}\right) = \arctan\left(\frac{-\gamma(1 - Q^2(1 - \gamma^2))}{Q}\right)$$



# Appendix #3. Fitting results.

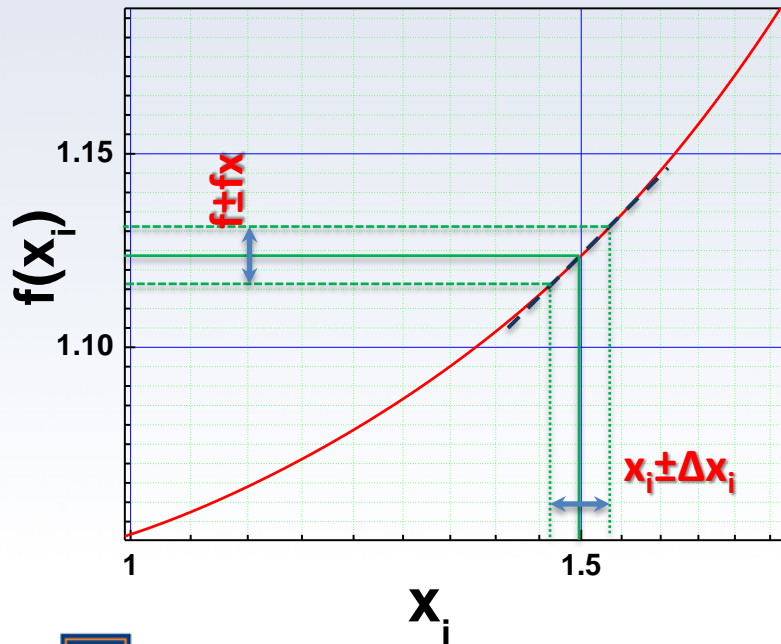


# Appendix #4.

# Error propagation.

$$y = f(x_1, x_2 \dots x_n)$$

$$\Delta f(x_i, \Delta x_i) = \sqrt{\sum_{i=1}^n \left[ \frac{\partial f}{\partial x_i} \right]^2 \cdot \Delta x_i^2}$$



# Error propagation. Example.

Derive resonance frequency  $f$   
from measured inductance  
 $L \pm \Delta L$  and capacitance  $C \pm \Delta C$

$$f_0(L, C) = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$$L_1 = 10 \pm 1 \text{mH}, \quad C_1 = 10 \pm 2 \mu\text{F}$$

$$\Delta f(L, C, \Delta L, \Delta C) = \sqrt{\left[\frac{\partial f}{\partial L}\right]^2 \cdot \Delta L^2 + \left[\frac{\partial f}{\partial C}\right]^2 \Delta C^2}$$

$$\frac{\partial f}{\partial L} = \frac{-1}{4\pi} C^{-\frac{1}{2}} L^{-\frac{3}{2}};$$

$$\frac{\partial f}{\partial C} = \frac{-1}{4\pi} L^{-\frac{1}{2}} C^{-\frac{3}{2}}$$

Results:

$$f(L_1, C_1) = 503.29212104487 \text{Hz}$$

$$\Delta f = 56.26977 \text{Hz}$$

$$f(L_1, C_1) = 503 \pm 56 \text{Hz}$$



# Error propagation. Example.

$L_1 = 10 \pm 1\text{mH}$ ,  $C_1 = 10 \pm 1\mu\text{F}$     Where these numbers are coming from?

1. Using commercial resistors, capacitors, inductances...



$C=500\text{pF}\pm 5\%$



$L=35\text{mH}\pm 10\%$

2. Measuring the parameters using standard equipment

**SENCORE "Z" meter model LC53**

Capacitance measuring accuracy  $\pm 5\%$

Inductance measuring accuracy  $\pm 2\%$



**Agilent E4980A Precision LCR Meter**

Basic accuracy  $\pm 0.05\%$

